Merging Asset Allocation and Longevity Insurance: An Optimal Perspective on Payout Annuities

by Peng Chen, Ph.D., and Moshe A. Milevsky, Ph.D.

Although several recent papers in the *Journal of Financial Planning* have discussed the mechanics and importance of lifetime, or payout annuities, the industry currently lacks a coherent and formal model of how much wealth should be allocated within and between these products. This paper revisits the importance of longevity insurance—while discussing the concerns with a strategy consisting purely of fixed payout annuities—and then addresses the proper asset allocation between conventional financial assets and payout annuity products. Our focus is on maximizing a suitably defined objective function in an intuitive, comprehensible and practical manner. In addition to the usual risk and return preferences of investors, our modeling framework requires inputs on the relative strength of retirees' bequest motives, their subjective versus objective health status, and their pre-existing longevity insurance (aka pensions). To illustrate the model, we provide some brief case studies.

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A number of recent articles in the *Journal of Financial Planning*—for example, Ameriks, Veres and Warshawsky (2001), Duff (2001), Bengen (2001) and Goodman (2002)—have focused attention on payout annuities[1] and their ability to hedge against longevity risk and reduce the probability of outliving wealth.

Indeed, the shift in retirement funding from professionally managed defined benefit plans to defined-contribution personal savings vehicles also implies that investors need to make their own decisions on what products should be used to generate income in retirement. Investors must consider two important risk factors when making these decisions:

1. Financial market risk—that is, volatility in the capital markets that induces portfolio values to fluctuate. If the market drops or corrections occur early during retirement, the portfolio may not be able to cushion the added stress of systematic withdrawals. This may make the portfolio unable to provide the necessary income for the desired lifestyle or it may simply run out of money too soon.
2. Longevity risk—that is, the risk of outliving one's portfolio. Life expectancies have been increasing and retirees should be aware of the substantial chances for a long retirement and plan accordingly. This risk is faced by every investor, especially those taking advantage of early retirement offers or those who have a family history of longevity.

Traditionally, asset allocation is determined by constructing efficient portfolios for various risk levels based on modern portfolio theory.[2] Then, based on the investor's risk tolerance, one of the efficient portfolios is chosen. Modern portfolio theory is widely accepted in the academic and finance industries as the primary tool for developing asset allocations. Its effectiveness is questionable, however, when dealing with asset allocations for individual investors in retirement, because it does not consider longevity risk and the portfolio's random time horizon. This a [illegible] ngevity insurance during retirement and establishes a framework to study the total [illegible] decision in retirement, which includes both conventional asset classes and im [illegible] ucts.

TABLE 1

The Conditional Probability of Survival at Age 65

To Age:	Single Female	Single Male	At Least One Member of a Couple
70	93.8%	92.0%	99.5%
75	84.4%	79.9%	96.9%
80	70.9%	62.7%	89.1%
85	52.8%	41.0%	72.2%
90	31.6%	19.6%	45.0%
95	13.4%	5.8%	18.4%

Source: Society of Actuaries RP-2000 table

Americans are [illegible] e. According to pension mortality tables, the probability that an individual retiri [illegible] 0 is over 70 percent for females and over 62 percent for males. When combined with [illegible] se, the odds reach nearly 90 percent that at least one spouse will live to 80, and a 7 [illegible] one spouse will live to age 85. For a broader sense of the potential longevity risk, [illegible] 65-year-old can expect to live.[3]

95	13.4%	5.8%	18.4%

For example, the probability that at least one spouse will reach age 70 is computed as follows: 1 – (1 – 0.938)*(1 – 0.920) = 99.5%. As the reader can see from the table, *longevity risk*—the risk of outliving one's resources—is substantial and is the main reason that we believe lifetime annuities (alternatively known as payout annuities) will grow in popularity.

Payout Annuity and Its Insurance Against Longevity Risk

There has been a substantial amount of recent literature on the topic of the costs and benefits of lifetime payout annuities, but space constraints prevent us from providing a comprehensive review. Roughly speaking—and with apologies to the authors—the relevant literature can be partitioned into three categories.

The first category consists of the theoretical insurance economics literature that investigates the equilibrium supply and demand of life annuities in the context of a complete market and utility-maximizing investors. This includes the classical work by Yaari (1965), as well as Richard (1975), Brugiavini (1993), Yagi and Nishigaki (1993), and Milevsky and Young (2002). Broadly speaking, their main conclusions are that life annuities should play a substantial role in a retiree's portfolio.

In the second category, the empirical annuity literature examines the actual pricing of these products and whether consumers are getting their money's worth. These include a sequence of papers by Brown, Warshawsky, Mitchell and Poterba (1999, 2000, 2001) in various combinations.

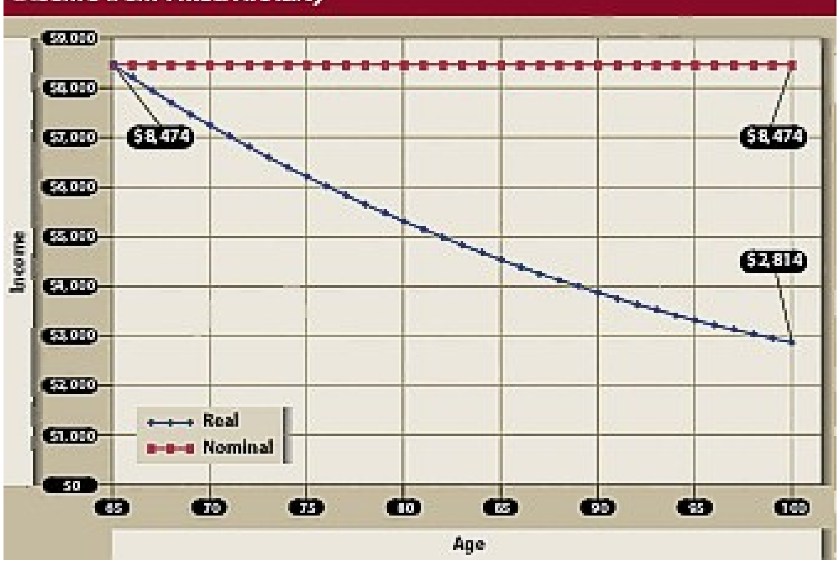
A third and final strand attempts to create normative models that help investors decide how much to annuitize, when to annuitize and the appropriate asset mix within annuities. These include the work by Ameriks, Veres and Warshawsky (2001), Milevsky (2000, 2001), Kapur and Orszag (1999), and Blake, Cairns and Dowd (2000).

Fixed Payout Annuity

Figure 1 illustrates the payment stream from a fixed immediate annuity. With an initial premium or purchase amount of $100,000, the annual income payments for a 65-year-old male in today's environment would be $706.14 a month, or $8,474 a year.[4] The straight line represents the annual payments before inflation. People who enjoy the security of a steady and predictable stream of income may find a fixed annuity appealing.

FIGURE 1

Income from Fixed Annuity



Despite the benefits of longevity insurance and fixed payout amounts, there are disadvantages with a portfolio that consists solely of fixed annuities. Because the payments are the same year after year, purchasing power is eroded as the annuitant gets older. The second curved line represents the same payment stream after a hypothetical 3.2 percent inflation rate is factored in.[5]

Figure 2 displays the inflation rate during the last 30 years, as measured by changes in the level of the Consumer Price Index (CPI). Notice that although the inflation rate in the United States is currently under two percent, this number is at the low end of the historical record. In fact, as recently as the early 1990s, the inflation rate was over 6 percent, and in the early 1980s, it went as high as 13 percent. The (arithmetic) average during the last 30 years was approximately five percent annually.

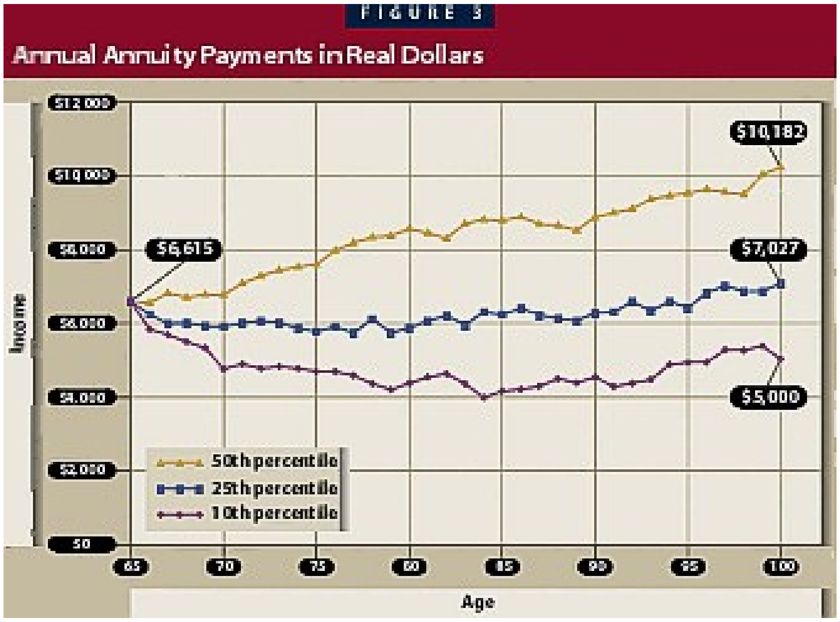
Besides the devastating impact of inflation on fixed payout annuities, a second concern is that investors cannot trade out of the fixed payout annuity once it is purchased.[6] In other words, the lack of liquidity (and reversibility) within a fixed product impedes the optimal asset allocation process and makes the fixed payout annuity less desirable, all else being equal. See Browne, Milevsky and Salisbury (2003) for details on how to quantify this drawback.

Finally, it seems that the current payout rates from fixed payout annuities are at a historical low, which is consistent with the current interest rate environment. A 65-year-old female might have received as much as $1,150 a month in the early 1980s, in exchange for the same $100,000 initial premium. Today that $100,000 buys closer to $700 a month. In fact, we are currently at historical lows on the interest rate cycle and this may be one of the worst times to lock in an interest for the rest of one's life. Recall that once the individual has purchased a life annuity, he or she can no longer cash in or sell the insurance contract. While we obviously want to refrain from speculating—and encouraging others to speculate—on the long-term direction of interest rates, we want to remind the reader that locking in a fixed annuity is implicitly a market timing play. This is why we believe that variable payout annuities can only grow in popularity.

Variable Payout Annuities

A variable payout annuity is an insurance product that converts an accumulated investment into a commitment by the insurance company to pay out annuity units over the lifetime of the investor. The annuity payments fluctuate in value depending on the underlying investments held and, therefore, disbursements will also fluctuate. Thus, instead of getting fixed annuity payments, the annuitant receives the equivalent of a fixed number of fund units. The insurance company converts these fund units into dollars at the going net asset value.

Figure 3 illustrates the annuity payment stream in real terms from a 50 percent stock/50 percent bond portfolio using a life-only payment option in an immediate variable annuity. We generated a Monte Carlo simulation to illustrate the various payment scenarios. The simulation is generated using historical return statistics of stocks, bonds and inflation from 1926–2001, a $100,000 initial portfolio and a three percent assumed investment return (AIR). While the actuarial mechanics behind the AIR are beyond the scope of this paper, one can think of it as a method of front-loading or back-loading annuity payments. The initial payment at age 65 is estimated to be $6,615.[7] To indicate a measure of the risk in the product, the three lines show the 10th, 25th and the 50th percentile respectively. Assuming that the annuitant survives to age 100, there is a 10 percent chance that annual inflation-adjusted annuity payments would have fallen below $5,000, a 25 percent chance that they would have fallen below $7,027 and a 50 percent chance that they would have grown to over $10,182.



Optimal Asset Allocation Mix with Payout Annuities

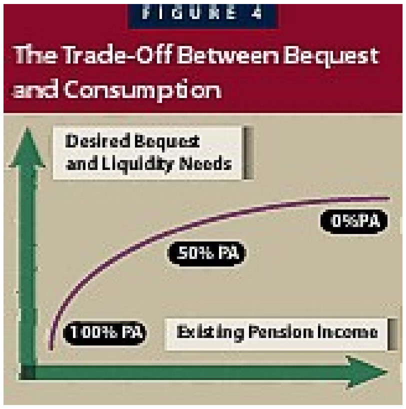
It makes little financial sense to offer a money market and bond fund in the savings portion of a personal pension plan, without offering an equity fund to complete the risk and return spectrum. Similarly, it makes little sense to offer fixed payout annuities without offering variable payout annuities to balance out the risk. Clearly, the latter is the symmetric extension of the former. And because there is a proper asset allocation involving savings (accumulation) products, the same applies to dissavings (consumption stage) products.

To determine the appropriate asset mix, classical asset allocation (savings) models used by the popular software vendors and advisor services enter information on the investor’s time horizon and risk aversion level. But to incorporate payout annuities and retirement dynamics into asset allocation models, a proper model requires more information. This would include inputs such as the investor’s subjective health estimate, the strength of bequest motives and pre-existing pension income.

We have developed a model for optimally allocating investment assets within and between two distinct categories. The two categories are annuitized assets and non-annuitized assets. The annuitized assets include fixed and variable immediate annuities. The non-annuitized assets include all types of investment instruments, such as mutual funds, stocks, bonds and T-bills that do not contain a mortality-contingent income flow. In addition, our model incorporates the following decision factors:Investor’s risk tolerance

- Investor’s age
- Investor’s subjective probability of survival
- Population objective (pricing) probability of survival
- Relative weights placed on consumption and bequest
- Investor’s utility from “live” consumption and bequest
- Risk and return characteristics of risky and risk-free assets

The model is developed based on micro-economic models of consumer behavior. The appendix provides a more technical discussion about the model. Figure 4 provides a graphical illustration of the trade-off between the desire for bequest and liquidity needs and existing pension income. The greater the desire for creating an estate, or bequest value, the lower the demand (or need for) payout annuities. This is because life annuities trade off longevity insurance against the creation of an estate.



Numerical Results

To understand the normative predictions of the model, let us look at three different cases so that we can see the effect of changing parameters on the optimal allocation.

We will start with the capital market assumptions that will remain the same for all four cases. All cases will assume that the individual is a 60-year-old male who would like to allocate his portfolio across the two investment asset classes and the two mortality-contingent claim classes. Together, the four “allocatable” products areRisk-free asset

1. Risky asset
2. Immediate fixed annuity
3. Immediate variable annuity

We assume that the return from the risk-free (T-bills) asset class is five percent annually with no volatility. The return from the risky asset is lognormally distributed with a mean value of 10 percent and a standard deviation of 20 percent. (In other words, the investment is expected to earn 10 percent annually, but may actually earn as much as 30 percent or lose 10 percent in any given year.) This implies a risk premium of five percent, which is in line with estimates for future U.S. equity markets. As for the mortality parameters, we use a table provided by the U.S.-based Society of Actuaries, called the Individual Annuity Mortality (IAM) 2000 basic table. These tables are the probabilities of survival for a healthy population of potential annuitants. Many people might feel they are less (or more) healthy than the numbers indicated by the IAM 2000; we will, therefore, allow the subjective probability of survival to be lower (or higher) than the objective probability of survival. The utility preferences will be taken from within the Constant Relative Risk Aversion (CRRA) family, with a CRRA coefficient of g.

While space prevents us from providing a crash course on micro-economic theory, the CRRA can be viewed as measuring a consumer’s aversion to investing in risky assets. The greater the CRRA value, the lower is their appetite for risk. And while we are fully cognizant that few if any investors can identify their personal CRRA value —and DNA testing has proved elusive so far—we strongly believe this normative framework can be used to guide a prudent asset mix and educate the investor about the risks.

Finally, we arbitrarily assume that the individual intends to hold this particular asset/product mix for 20 years.[8] In practice, we would recommend that investors rebalance their portfolio much *before* the 20- year horizon, which requires a dynamic multi-period model. This additional dimension of when to rebalance is beyond the scope of this introductory paper and is being addressed in a follow-up paper by Peng and Milevsky (2003), as well as the theoretical model developed by Milevsky and Young (2002).

Case #1: Total Altruism and Complete Bequest Motives

In this case, we assume the investor's utility is derived entirely from bequests. In other words, the weight of his utility of bequest is assumed to be one and the weight on his utility of consumption is zero—that is, A = 0 and D = 1. The objective probability of survival is 65 percent (roughly equal to the survival probability of a 60-year-old male in the next 20 years) and the subjective probability is the same 65 percent. This assumes that the investor does not have any private information about his or her mortality status that might lead them to believe they are healthier, or less healthy, than the average annuitant. Using these input parameters in the model described above, the optimal allocations to the assets across various relative risk aversion levels are presented in Table 2 and Figure 5.

TABLE 2

Optimal Allocations

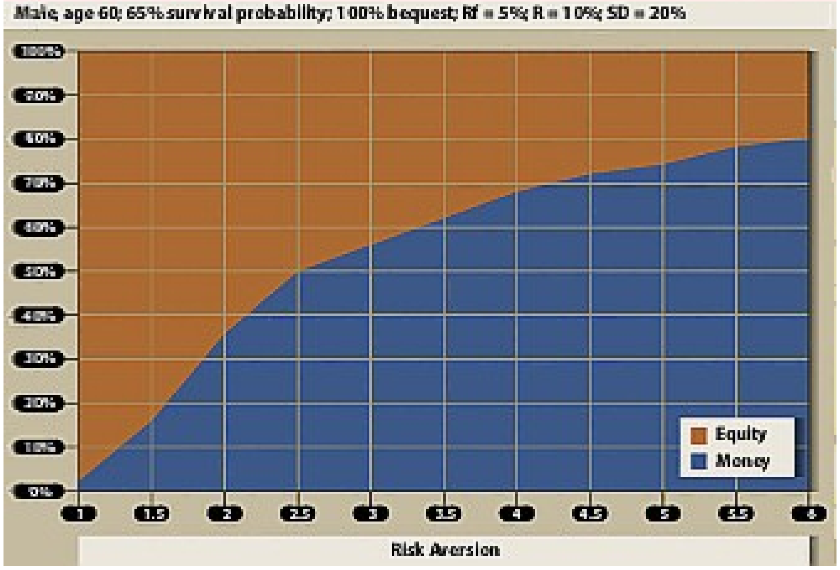
- Male age 60 with a 100% bequest motive
- 20-year horizon in an economy with a risk-free rate = 5%
- Expected return from risky (equity market) asset = 10%
- Standard deviation of investment return = 20%

Risk Aversion	Money	Equity	FIA	VIA	Total Risk-Free	Total Risky	Total Conventional	Total Annuity
1	2%	98%	0%	0%	2%	98%	100%	0%
1.5	16%	84%	0%	0%	16%	84%	100%	0%
2	36%	64%	0%	0%	36%	64%	100%	0%
2.5	50%	50%	0%	0%	50%	50%	100%	0%
3	56%	44%	0%	0%	56%	44%	100%	0%
3.5	62%	38%	0%	0%	62%	38%	100%	0%
4	68%	32%	0%	0%	68%	32%	100%	0%
4.5	72%	28%	0%	0%	72%	28%	100%	0%
5	74%	26%	0%	0%	74%	26%	100%	0%
5.5	78%	22%	0%	0%	78%	22%	100%	0%
6	80%	20%	0%	0%	80%	20%	100%	0%

FIGURE 5

Optimal Allocations

Male, age 60; 65% survival probability; 100% bequest; Rf = 5%; R = 10%; SD = 20%



A few things should be evident from the table. First, immediate annuities get no allocation because the investor cares only about the bequest. The intuition for this result can be traced back to a classical paper by Yaari (1965). Namely, if consumers are 100 percent altruistic, they will not waste the asset by annuitizing. Second, the allocation to stocks gradually decreases as the investor's risk aversion increases. Thus, without any consumption motive, this becomes the traditional allocation problem between risk-free and risky assets. This case can be used as an illustration for very wealthy individuals, where the size of their portfolio far exceeds their consumption needs. In this case, bequest becomes the dominant factor. Annuities do not get any allocation, as they do not leave any money for the heirs. For example, for investors with a relative risk aversion level of 2, the optimal allocation is 36 percent to the risk-free asset and 64 percent to equity.

Case #2: No Bequest Motives

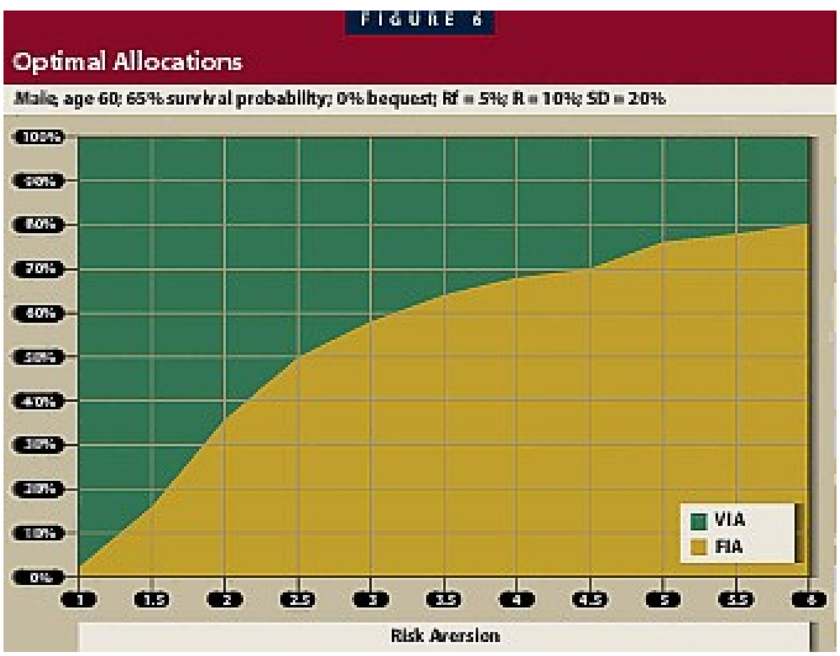
This case maintains the same age, gender, survival probability and time horizon, but completely eliminates the bequest by replacing A = 1 with D = 0. In other words, 100 percent of the utility weight is placed on “live” consumption. The optimal allocations to the assets across various risk aversion levels are presented in Table 3 and Figure 6.

TABLE 3

Optimal Allocations

Same as Table 2, but with 0% bequest motive

Risk Aversion	Money	Equity	FIA	VIA	Total Risk-Free	Total Risky	Total Conventional	Total Annuity
1	0%	0%	2%	98%	2%	98%	0%	100%
1.5	0%	0%	16%	84%	16%	84%	0%	100%
2	0%	0%	36%	64%	36%	64%	0%	100%
2.5	0%	0%	50%	50%	50%	50%	0%	100%
3	0%	0%	58%	42%	58%	42%	0%	100%
3.5	0%	0%	64%	36%	64%	36%	0%	100%
4	0%	0%	68%	32%	68%	32%	0%	100%
4.5	0%	0%	70%	30%	70%	30%	0%	100%
5	0%	0%	76%	24%	76%	24%	0%	100%
5.5	0%	0%	78%	22%	78%	22%	0%	100%
6	0%	0%	80%	20%	80%	20%	0%	100%



Because the returns on annuities are always higher than the returns on traditional assets—conditional on the retiree being alive—the immediate annuities get 100 percent of the allocation. The allocation to the immediate variable annuity gradually decreases, while the allocation to the immediate fixed annuity increases as the risk aversion of the investor increases. This case can be used as another extreme illustration for investors who would like to maximize their lifetime consumption and have no interest in leaving any bequest or estate. (They are alternatively known as the “die broke” crowd.) All the savings should be used to purchase annuities. Overall, the optimal allocation between risky and risk-free assets (in this case, an immediate fixed annuity and an immediate variable annuity) are almost identical to that of Case #1. For investors with a risk aversion level of 2, the optimal allocation is 36 percent to immediate fixed annuity and 64 percent to immediate variable annuity.

Case #3: 20 Percent Bequest Motives and 80 Percent Consumption Motives

This case maintains the same age, gender, survival probability and time horizon, but changes the strength of bequest from D = 0 to a more realistic D = 0.2.[9] In other words, 80 percent of the utility weight is placed on "live" consumption. The optimal allocations to the assets across various risk aversion levels are presented in Table 4 and Figure 7.

TABLE 4

Optimal Allocations

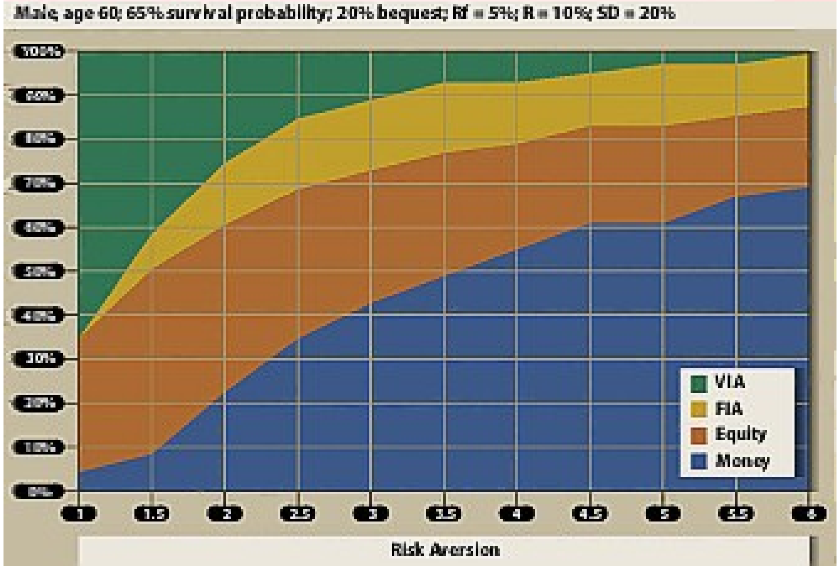
Same as Table 2, but with 20% bequest motive

Risk Aversion	Money	Equity	FIA	VIA	Total Risk-Free	Total Risky	Total Conventional	Total Annuity
1	4%	30%	0%	66%	4%	96%	34%	66%
1.5	8%	42%	8%	42%	16%	84%	50%	50%
2	22%	38%	14%	26%	36%	64%	60%	40%
2.5	34%	34%	16%	16%	50%	50%	68%	32%
3	42%	30%	16%	12%	58%	42%	72%	28%
3.5	48%	28%	16%	8%	64%	36%	76%	24%
4	54%	24%	14%	8%	68%	32%	78%	22%
4.5	60%	22%	12%	6%	72%	28%	82%	18%
5	60%	22%	14%	4%	74%	26%	82%	18%
5.5	66%	18%	12%	4%	78%	22%	84%	16%
6	68%	18%	12%	2%	80%	20%	86%	14%

FIGURE 7

Optimal Allocations

Male, age 60; 65% survival probability; 20% bequest; Rf = 5%; R = 10%; SD = 20%



There are several interesting results in the allocation. First, unlike the previous two cases, all four of the asset classes are present in the optimal allocations. This is because immediate annuities are more suitable (relative to traditional assets) for consumption and traditional investments are better suited for satisfying bequest motives. When the investor has a more balanced motive between bequest and consumption, both immediate annuities and traditional asset classes are selected. In general, the higher the bequest motives, the more the investor should allocate to traditional investments and the less to payout annuities.

Second, the allocation between risky (both variable income annuities and equity) and risk-free (cash and fixed income annuities) is almost identical to that in Case #1 and Case #2 at comparable risk aversion levels. This indicates that the changes in the investor's bequest versus consumption motive do not significantly affect the investor's behavior regarding risk. The optimal allocation between risky and risk-free assets is determined by the investor's risk tolerance.

Third, we find that the allocation to annuities decreases as the investor's risk aversion level increases. In other words, extremely risk-averse investors will actually avoid payout life annuities. At first this result might seem counter-intuitive because higher risk aversion is normally associated with more (not less) insurance. But one can rationalize this result by remembering that individuals with some preference for bequest get little or no utility from a payout annuity if they die shortly after the purchase. This bequest-loss "risk" might drive the optimal allocation away from annuities, which is similar to the impact of a higher explicit weight on the utility of bequest.

Summary and Conclusions

Motivated by the recent interest on the topic of annuitization within the public debate about pension provision, this paper has investigated the theory and practice of constructing an optimal asset allocation during retirement. Our model differs from some of the previous work by considering *both* financial market risk *and* longevity risk in the portfolio trade-off analysis.

Our main qualitative insight is shown in Table 5. The natural asset allocation spectrum consists of investments that go from safe (fixed) to risky (variable). In contrast, the product allocation spectrum ranges from conventional savings vehicles to annuitized payout (pension) instruments. The asset and product spaces are separate dimensions of a well-balanced financial portfolio, yet both must be addressed at retirement.

TABLE 5

Asset Allocation Spectrum

Exhibit	Product Allocation	
Asset Allocation	Conventional	Annuitized
	Fixed (CDs, T-Bills, Bonds)	Fixed (Fixed Payout Annuity)
	Variable (Stocks, Equity Mutual Funds)	Variable (Variable Pay-out Annuity)

More formally, we have presented a mathematical one-period model to analyze the optimal allocations within and between payout annuities. The numerical results confirm that the optimal allocations across assets and products are influenced by many factors, including age, risk aversion, subjective probability of survival, utility of bequest, and the expected risk and return trade-offs of different investments. We also find that the global allocation between risky and risk-free assets is influenced only by the investor's risk tolerance; it is not significantly affected by the subjective probability of survival or the utility of consumption versus bequest.

Thus, in some sense we are advocating a classical economic "separation theorem" argument. We claim that the first step of a well-balanced retirement plan is to locate a suitable mix of risky and risk-free assets independently of their mortality-contingent status. Then, once a comfortable balance has been struck between risk and return, the annuitization decision should be viewed as a second-step "overlay" that is placed on top of the existing asset mix. And, depending on the strength of bequest motives and subjective health assessments, the optimal annuitized fraction will follow.

Of course, retirement is not just one point or period in time. Ongoing research by the authors is partitioning the golden years into various stages to examine optimal *dynamic* policies as one moves through and toward the end of the life cycle.

Endnotes

1. We are careful to distinguish between the term *variable annuity*—which is currently (mis)used by the industry to promote a savings product—and a payout annuity which is a disbursement product or phase where the insurance company guarantees an income stream that cannot be outlived.
2. Markowitz (1952) and Merton (1971).
3. We have chosen age 65 as the standard baseline for retirement, although similar numbers can be generated for any age.
4. This is the average quote obtained by the authors in mid-July 2002, assuming a 65-year-old male and a $100,000 premium. The payments from different companies can differ quite substantially from week to week and from the best to the worst insurance company quotes.
5. The average inflation rate from 1926 to 2001 was 3.2 percent in the United States.
6. There are payout annuities available that allow the investor to withdraw money, but the investor typically has to pay a surrender or market value adjustment charge. Furthermore, this would only apply during the certain period of the annuity where payments are guaranteed regardless of life status.
7. The initial payment is estimated by Ibbotson Associates.
8. These assumptions can be easily modified to accommodate other utility functions, asset return distributions, mortality probabilities and horizons. Note that because we are using a utility function that has constant relative risk aversion the initial wealth level does not have any impact on the allocations for the one-period model.
9. See Bernheim (1991), Hurd (1989), as well as Abel and Warshawsky (1988) for a discussion and estimates of the "strength of bequest" parameters. We have taken 20 percent as an approximation.

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Appendix: Technical Model of Optimal Asset Allocation

In this technical appendix, which we aim toward the braver readers, we present the formal mathematical model that underlies the numerical results in the body of paper. We start by assuming that a rational investor is choosing the allocations of his or her retirement portfolio to maximize a well-defined utility function. We also assume that there are only four different products to choose from: (1) risk-free asset, (2) risky asset, (3) immediate fixed payout annuity and (4) immediate variable payout annuity. We can easily expand this model to incorporate more asset classes.

The category matrix presented in Table A-1 summarizes the returns from the four possible products, conditional on being alive or dead.

TABLE A-1

The Four Basic Investment Products

	Alive	Dead
Risk-Free Asset (T-bills):	**R**	**R**
Risky Asset (Equity):	**X**	**X**
Immediate Fixed Annuity:	(1+**R**)/p	0
Immediate Variable Annuity:	(1+**X**)/p	0

From a mathematical point of view, we have the following problem. We are looking for asset allocation weights, denoted by $\{a_1, a_2, a_3, a_4\}$ that maximize the objective function:

$$E[U(W)] = \bar{p} \times A \times E[u(a_1 wR + a_2 wX + a_3 wR / p + a_4 wX / p)]$$
$$+ (1 - \bar{p}) \times D \times E[u(a_1 wR + a_2 wX)]$$
$$S.T.$$
$$a_1 + a_2 + a_3 + a_4 = 1$$
$$a_i > 0$$

where we use the following notation:

- The letter A denotes the relative strength placed on the utility of consumption.
- The letter D denotes the relative strength placed on the utility of bequest. The sum of A and D are assumed to be one, so there is only one free variable. Individuals with no utility of bequest will be assumed to have D = 0.
- The symbol p denotes the objective probability of survival, which is the probability that is used by the insurance company to price immediate annuities.
- The symbol $\bar{p}$ denotes the subjective probabilities of survival. The subjective probability of survival may not match the objective population (annuitant) probability. In other words, a person might believe he or she is healthier (or less healthy) than average. This would affect the expected utility but not the payout from the annuity, which is based on objective (annuitant) population survival rates.
- The bold letter **X** denotes the (one plus) random return from the risky asset and the letter **R** denotes the (one plus) risk-free rate.
- The expression $E[u(a_1 w\mathbf{R} + a_2 wX + a_3 w\mathbf{R}/p + a_4 w\mathbf{X}/p)]$ denotes the utility from the live stage, while $E[u(a_1 w\mathbf{R} + a_2 w\mathbf{X})]$ denotes utility from the dead state. Notice that the annuity term, which divides by the probability of survival, does not appear in the dead state. This is because the annuity does not pay out.
- The function u(.) denotes the standard utility function of end-of-period wealth. Our model can handle cases of both constant relative risk aversion and decreasing relative risk aversion, as well as other functional forms that are consistent with loss aversion.

Because the weights $\{a_1, a_2, a_3, a_4\}$ sum to one, we essentially have only three variables in the search process. An important factor to consider in solving the utility maximization is that as functions of (a_1, a_2, a_3), both E[U(W)] and its derivatives are defined by integrals that must be evaluated numerically.